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Some Comments on $N = 2$ Supersymmetric Yang-Mills

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We comment on some aspects of the semiclassical BPS saturated states close to the curve $\mathrm{Im}\, a_D/a = 0$.

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Recently Seiberg and Witten have given an exact description of the low-energy $N = 2$ Yang-Mills theory [1]. They have furthermore determined the exact mass spectrum of the BPS saturated states in the theory. More precisely if one knows the quantum numbers (in this case the magnetic and electric charges) of a particle which exists then one can calculate its exact mass. The spectrum of states in the semi-classical limit is known to consist of the $N = 2$ vector multiplets with the photon and electrically charged Ws as their spin 1 components. There are also solitons carrying magnetic charge 1 and arbitrary integral electric charge. The magnetically charged solitons form spin $\leq 1/2$ hypermultiplets [2]. All the semiclassical states saturate the so-called BPS bound. The quantum theory has a space of vacua which is parameterized by $u = <tr\phi^2>$, where ϕ is the scalar component of the $N = 2$ vector multiplet. The low-energy theory is $N = 2$ supersymmetric QED almost everywhere: at two points (which can be taken to be $u = -1, 1$) extra particles become massless, these particles have spin $\leq 1/2$ and so the $SU(2)$ gauge symmetry is never restored. As argued in [1] the semi-classical BPS saturated states continue to exist as long as one does not cross a curve on which a_D/a is real. When such a curve is crossed some of the states which exist semi-classically become degenerate with some multi-particle stable BPS states. In this note we demonstrate : 1) that there is a curve passing through $u = -1, 1$ on which a_D/a is real and that this curve is diffeomorphic to a circle; 2) that all the states except for the monopoles/dyons becoming massless at $u = -1, 1$ become degenerate with multiparticle states consisting of only these two particles on the $\text{Im}a_D/a = 0$ curve. Further, assuming that the only semi-classically massive BPS saturated particles which continue to exist are these two distinguished states, then the monodromies naturally work out.

We first demonstrate that there is a curve diffeomorphic to the circle on which a_D/a is real. Notice that this condition is an SL(2,Z) invariant statement, so it is natural to find an SL(2,Z) invariant *function* on the u plane which takes a particular value when and only when this condition is satisfied. The Kähler potential

$$K = \text{Im}\bar{a}a_D$$

is such a function. $K = 0$ if and only if $\text{Im}a_D/a = 0$. The manifold we are interested in is then $N = K^{-1}(0)$. If we can show that 0 is not a critical value of K then we will have shown that N is a smooth one dimensional manifold [3]. Once we have established this fact we need only check that the $K = 0$ curves passing through $u = -1, 1$ are not disjoint and do not continue to infinity.

We begin by establishing that 0 is not a critical value of K. Let us fix a point $u \neq -1, 1$. Then $dK = 0$ and $K = 0$ are both satisfied only if $\mathrm{Im}\partial_u a_D / \partial_u a = 0$. However, we know that K is a good Kähler potential everywhere except at $u = -1, 1$, therefore this condition is in conflict with a positive metric and cannot be satisfied away from the points $u = -1, 1$. Now let us take $u = 1$, then using the explicit expressions calculated in [1] we know that $dK = 1/\pi (du + d\bar{u}) \neq 0$ at $u = 1$. Similarly $dK \neq 0$ at $u = -1$ either, this follows from the Z_2 symmetry of the theory. Explicitly, $a(u) = -i4/\pi + i/2\pi (u + 1) \ln(u + 1)$ and $a_D + a = 1/2(u + 1) + O(u + 1)^2$ close to $u = -1$. Therefore we see that $K^{-1}(0)$ is a smooth one dimensional manifold [3].

We can now eliminate the possibility that either one of the $K = 0$ curves passing through $u = -1, 1$ continue to infinity. This follows from the expressions evaluated in [1] for $\mid u \mid \gg 1$: $K \approx 2/\pi \mid u \mid \ln \mid u \mid > 0$. There are now only two possible behaviors[1] for the $K = 0$ curves passing through $u = -1, 1$: 1)that they form two disjoint closed curves or 2)that there is a curve diffeomorphic to the circle passing through $u = -1, 1$. The first scenario is easily seen not to be realized from the expressions for a_D, a in terms of integrals given in [1] , one sees immediately that for u real and $-1 < u < 1$, $\mathrm{Re}a_D = 0, \mathrm{Im}a_D \neq 0$ and $\mathrm{Re}a \neq 0, \mathrm{Im}a \neq 0$. Similarly for u real and $u < -1$ we have $\mathrm{Im}a_D \neq 0, \mathrm{Re}a_D \neq 0$ and $\mathrm{Im}a \neq 0, \mathrm{Re}a = 0$. The situation for u real and $u > 1$ has already been discussed in [1] where it was shown that $K \neq 0$ in that region. So the only points on the real axis where $K = 0$ are $u = -1, 1$. This establishes that there is a curve diffeomorphic to S^1 passing through $u = -1, 1$. An analytic argument for the existence of this curve has also been given in[2][4].

Since the manifold $K^{-1}(0)$ consists in general of (possibly) many disjoint one dimensional manifolds diffeomorphic to S^1 we shall denote by $\tilde{N}$ the particular one passing through the points $u = -1, 1$. We would now like to discuss the physical spectrum as we cross the curve $\tilde{N}$. We begin by establishing some general features. Let us first recall that according to [1] there are only two points on the u-plane at which two semiclassically massive states become massless. This implies the following: a_D/a is an integer only at $u = -1, 1$, since if at some point $a_D/a = n$ then the state $(1, -n)$ becomes massless at

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[1] We are making use of the fact that at $u = -1, 1$ the real axis is not tangent to the curve $K = 0$. This has been shown for $u = 1$ in [1] and for $u = -1$ follows from the Z_2 symmetry of the theory and from the explicit expressions given for a_D, a in the previous paragraph.

[2] I would like to thank J. de Boer for informing of this reference.

this point. Therefore, on the curve $\tilde{N}$ a_D/a is non-integer except at the points $u = -1, 1$ where it is integral. The values of a_D/a at these two points determine in which range the values of a_D/a will take. Let us consider a monodromy around a point $u = u_0$ with a base point in the semi-classical region ($\mid u \mid >> 1$) and located in the u upper-half plane. If the particle becoming massless at $u = u_0$ has charge vector $(1, n)$ then the monodromy matrix for an anti-clockwise path around u_0 is given by:

$$M_{u_0} = \begin{pmatrix} 1 + 2n & 2n^2 \\ -2 & 1 - 2n \end{pmatrix}$$

. If we restrict ourselves to a region including the base point and u_0 but no other singular points. Then in this region $a_D + na$ is a good coordinate and has a Taylor expansion close to u_0. Although a does not have a Taylor expansion, it can be expanded in fractional powers of $(u - u_0)$ and $(u - u_0)^k \ln(u - u_0)$. Specifically,

$$a_D + na \approx c(u - u_0) + o(u - u_0)^2$$

$$a \approx b + i\frac{c}{\pi}(u - u_0)\ln(u - u_0)$$

where b, c are constants. If we also assume that $\tilde{N}$ looks like $u = u_0 + it$ for t real close to $u = u_0$, then $\text{Re}c/b = 0$. We come to the main point of this detour which is to fix the relative sign of a_D, a which individually do not have well defined signs. The relative sign appears in the masses of the BPS saturated states and is therefore physical. The curve $\tilde{N}$ is a smooth manifold and we know that there is a path to u_0 from the semi-classical region along which $\text{Im}a_D/a > 0$, in particular approaching $u = -1(1)$ from $-\infty(+\infty)$ along the real axis never crosses a point where $\text{Im}a_D/a = 0$. $\tilde{N}$ defines a region which is essentially the strong coupling region. Immediately outside (resp. inside) of this region $\text{Im}a_D/a > 0$ (resp. $\text{Im}a_D/a < 0$). This fixes the sign of $\text{Im}c/b$ as follows, if $u_0 = -1$ then $\text{Im}c/b < 0$ and if $u_0 = 1$ then $\text{Im}c/b > 0$. Close to $u = u_0$,

$$a_D/a \approx -n + \frac{c}{b}(u - u_0),$$

so a_D/a increases if one follows $\tilde{N}$ in a clockwise direction and decreases in the anti-clockwise direction. In fact one can say more: a_D/a is monotonic on $\tilde{N}$. This can be established by noting that the derivative of a_D/a is always non-zero on $\tilde{N}$ if the metric is positive everywhere on $\tilde{N}$ (excluding the two singularities) and assuming that a never becomes infinite.

Before proceeding to the specifics, we note a potentially confusing point. At $u = u_0$ the lowest mass particle is the one becoming massless at that point, i.e. a particle with charge $(1, n)$. The particles with the next highest mass are the ones with charges $(1, n-1), (1, n+1), (0, 1)$. These three particles are degenerate in mass, the first two are dyons and the last the W. As we move away from $u = u_0$ the degeneracy in mass is lifted, however the identity of the lightest particles is different depending on whether one moves clockwise or anti-clockwise from u_0 on $\tilde{N}$. While $(1, n)$ always remains one of the two lightest particles, the other light particle is $(1, 1+n)$ (resp. $(1, n-1)$) if one moves anti-clockwise (resp. clockwise) away from u_0 on $\tilde{N}$. If one follows the curve $\tilde{N}$ towards the singularity lying outside of the region enclosed by the path defining the monodromy one will see different particles becoming massless at this point. This is consistent since the identity in terms of charge vectors of the particle becoming massless at the other singularity is not invariant under the monodromy M_{u_0}. However, presently we will propose that only the two lightest dyons be present inside the strong coupling regime and we will label them differently depending on whether we are considering a clockwise or anti-clockwise path around u_0. That this proposal is plausible can be seen as follows. If one asks using the monodromy M_{u_0} how many particles come back with a mass equal to that of a semi-classical state the answer is precisely two. These two states are $(1, n)$ and $(1, n+1)$ for the anti-clockwise monodromies and $(1, n)$ and $(1, n-1)$ for the clockwise monodromies.

Consider a closed path based at a point located at $| u | >> 1$ (i.e. in the semi-classical region) with $\mathrm{Im} u > 0$ which wraps once around the point $u = 1$. The path is homotopic to a path which crosses $\tilde{N}$ once in the upper and once in the lower-half planes. As the path crosses $\tilde{N}$ in the upper half plane $-1 < a_D/a < 0$ as shown above. The lowest mass particles at the point where the path crosses $\tilde{N}$ are $(1, 0), (1, 1)$, with masses $\sqrt{2} a_D$ and $\sqrt{2}(| a | - | a_D |)$. All the other semi-classical states have masses and charges degenerate with a multi-particle state consisting solely of these two particles. In particular, the Ws with charge vectors $(0, \pm 1)$ have the same mass as the sum of the masses of the particles $(\mp 1, 0)$ and $(\pm 1, \pm 1)$. Since the charges also add up the Ws must be degenerate with a two particle state consisting of these two particles. Similarly, the state $(1, n)$ with $n \geq 2$ has mass equal to the sum of the masses of $n - 1$ monopoles with charge $(-1, 0)$ and n dyons with charge $(1, 1)$. Again since the charges add up the state $(1, n)$ is degenerate with a $2n - 1$ particle state consisting solely of the two particles becoming massless at $u = -1, 1$. The remaining states with charge vector $(1, -n)$ with $n \geq 1$ are degenerate with a $2n+1$ multiparticle state consisting of n dyons of charge $(-1, -1)$ and $n+1$ monopoles of

charge $(1, 0)$. Since the states $(1, 1), (1, 0)$ do not become unstable as one enters the strong coupling regime there must exist one-particle states with these quantum numbers. The other semi-classical dyons are related to these two states by an M_∞ transformation. Since an M_∞ monodromy based at a point in the strong coupling regime always has to cross $\tilde{N}$, it is not necessary for the other dyons to exist as one-particle states inside the strong coupling regime. Also, the W need not exist either. If we assume this rather "maximal" scenario (i.e. the only semi-classical states which survive in the strong coupling regime are the particles becoming massless at $u = -1, 1$), then as we cross the $\tilde{N}$ in the lower half plane all the semi-classical states return as multi-particle states except for $(1, 1)$ which transforms to $(-1, 1)$ and $(1, 0)$ which returns untransformed.

Similarly, a clockwise path will be homotopic to a path which crosses the $\tilde{N}$ in the lower half plane first. At the intersection all the particles except $(1, -1)$ and $(1, 0)$ will become degenerate with a multiparticle state as follows:

$$(0, 1) \rightarrow (-1, 1) + (1, 0)$$
$$(1, n) \rightarrow n(-1, 1) + (n + 1)(1, 0) \text{ (with } n \geq 1)$$
$$(1, -n) \rightarrow n(1, -1) + (n - 1)(-1, 0) \text{ (with } n \geq 2)$$

If we again assume that the only semi-classical one-particle states in the strong coupling region are $(1, 0)$ and $(1, -1)$ (recall that the identity of the stable particles is not the same for the anti-clockwise and clockwise mondromies), then all the states except for these two come back as multiparticle semi-classical states. The state $(1, 0)$ is not transformed but $(-1, 1)$ comes back as a $(1, 1)$ dyon.

The monodoromies around $u = -1$ can be studied in a similar way. The only difference being that the identity of the stable particles in terms of charge vectors will be different from the $u = 1$ monodromies. In particular, the semi-classical particles which will return as one particle states are $(1, 1)$ and $(1, 2)$ $((1, 0))$ for the anti-clockwise (clockwise) monodromies.

In a recent paper [5] Lindström and Rôcek have noted that the W bosons have a negative kinetic term if one continues the semi-classically correct effective action to strong coupling. It would be interesting to devise methods of probing the spectrum in the stong coupling regime to establish whether the scenario presented here is infact realized.

The existence of the curve $\text{Im} a_D/a = 0$ has been established numerically in [6]

Acknowledgements

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